1 a) 1) $x_{n+1} = x_n - \frac{\sin(x_n)}{\cos(x_n)}$

2) $x_0 = 3$

$x_1 = 3{,}14254654$

$x_2 = 3{,}14159265$

b) 1) $g(x) = x + 2\sin(x)$

$g'(x) = 1 + 2\cos(x)$

$g'(\pi) = -1 \quad g'(x) \gtrsim -1$ if $x \approx \pi$

$\Rightarrow$ (very slow) convergence

2) $K = \frac{x_{1004} - x_{1003}}{x_{1003} - x_{1002}} = -0{,}99953678\,54$

$\varepsilon \leq \frac{K}{1-K} |x_{1004} - x_{1003}| = 0.03725\,7032$

3) $\hat{x}_n = x_n - \frac{(x_{n-1} - x_n)^2}{x_{n-2} - 2x_{n-1} + x_n}$

$= 3.14159265\,2$

4) $g(x) = x + \alpha \sin(x)$

$g'(x) = 1 + \alpha \cos(x)$

$g'(\pi) = 1 - \alpha = 0 \quad \Rightarrow \alpha = 1$

c) 1 initialisation

2 loop construction

3 iteration formula

4 error estimate in stop crit.: $|f(x_{n+1})| < tol$ OR $|x_{n+1} - x_n| < tol$

5 efficiency: use stored function values

2 a) 1) $f(m)\,\Delta x = \dfrac{\sin\left(\sqrt{\frac{\pi^2}{4}}\right)}{\sqrt{4 \cdot \frac{\pi^2}{4}}} \cdot \dfrac{\pi^2}{8} = \dfrac{\pi}{8}$

2) f singular at $x=0$
f not defined at $x=0$
$g''(0) = \infty$
interval $[0, \pi^2]$ not in full accuracy
Δx not in full accuracy

b) 1) $q = \left|\dfrac{I_{32} - I_{64}}{I_{64} - I_{128}}\right| = 4,00112887 43$
$\hat{=} 2^2$ so quadratic convergence

$\varepsilon_{128} = \dfrac{I_{128} - I_{64}}{3} = -5.0203333\ 10^{-5}$

$\varepsilon_{256} \approx \varepsilon_{128}/4 = -1.25508333\ 10^{-5}$

2) $\varepsilon_{128} = \dfrac{1-0}{24}\left(\dfrac{1}{128}\right)^2 M$

$f(x) = \pi \sin(\pi x) \Rightarrow f''(x) = -\pi^3 \sin(\pi x)$

$\Rightarrow M = \pi^3$

$\varepsilon_{128} = 7.885303\,9246\ 10^{-5}$

3) ε_{128} from 1) better because global theorem uses same M for whole interval

4) $T_2(128) = \frac{4}{3} I(128) - \frac{1}{3} I(64) = 1,9999999$
$T_2(64) = \frac{4}{3} I(64) - \frac{1}{3} I(32) = 1,9999594$
$T_3(128) = \frac{16}{15} T_2(128) - \frac{1}{15} T_2(64) = 2,000000006$

c) 1 grid
2 loop construction
3 trapezium formula
4 refinement procedure
5 error estimate (coeff $\frac{4}{3}$)
6 efficiency : re-use function values

3 a) 1) $f(x,y) = -y^2 + (2x+1)$

$k_1 = \frac{1}{2} f(0,1) = 0$

$k_2 = \frac{1}{2} f(\frac{1}{2}, 1+0) = \frac{1}{2}$

$y_{0.5} = 1 + \frac{1}{2}(0 + \frac{1}{2}) = 5/4$

$k_1 = \frac{1}{2} f(\frac{1}{2}, \frac{5}{4}) = \frac{1}{2} \times \frac{7}{16} = 0.21875$

$k_2 = \frac{1}{2} f(1, \underbrace{\frac{5}{4} + \frac{7}{32}}_{=\frac{47}{32}}) = 0.4213867187 5$

$y_{1.0} = \frac{5}{4} + \frac{1}{2}(0.21875 + 0.42138671875)$

$= 1.570067$

2) $y_1 = \underbrace{y_0}_{1} + \underbrace{\Delta x}_{1}(-y_1^2 + (2\underbrace{x_1}_{=1} + 1))$

$y_1^2 + y_1 - 4 = 0 \qquad y_1 = \frac{-1 \pm \sqrt{1+16}}{2}$

$y_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{17}$

(reject $y_1 = -\frac{1}{2} - \frac{1}{2}\sqrt{17}$)

b) 1) $\mathcal{E} = \dfrac{2.94122943 - 2.93886790}{3} = 8.53084333\ 10^{-4}$

2) $\dfrac{\mathcal{E}}{4^n} < 10^{-8}$

$n > 8$

grid $\Delta x = 0.125$ must be halved 9 times

$\Delta x \to \dfrac{0.125}{512} \approx 2.44\ 10^{-4}$

3) Extrap $= 2.94122943 + \mathcal{E} = 2.94208327\!\!\!/33$

c) $x=4$ $\left| \dfrac{I_{0.5} - I_{0.25}}{I_{0.25} - I_{0.125}} \right| \approx 113,9$

$x=5$ $\approx 975,4$

9 differs from 4 largely, computation with $\Delta x = 0.5$ is unstable (wiggle visible in column)

d) 1 grid
2 loop construction
3 RK2 formulas
4 error estimate in stop criterion
5 refinement procedure full grid

4 a) $\dfrac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + 2y_i = \cos(\pi x_i)$

$x_1 = 0 \quad x_2 = 0.1 \quad \cdots \quad x_{11} = 1$

$\Delta x = 0.1$

$N = 10$ segments

$N+1 = 11$ points

$$\begin{bmatrix} 1 & & & & \\ L & D & R & & \\ & L & D & R & \\ & & \ddots & & \\ & & L & D & R \\ & & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{10} \\ y_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ \cos(0.1\pi) \\ \cos(0.2\pi) \\ \vdots \\ \cos(0.9\pi) \\ 0 \end{bmatrix}$$

$L = 1/\Delta x^2 \quad R = 1/\Delta x^2 \quad D = -2/\Delta x^2 + 2$

11×11 matrix, 11-vector

b) include $3y'$ in ode $\rightarrow$ (1) $3\dfrac{y_i - y_{i-1}}{\Delta x}$

OR (2) $3\dfrac{y_{i+1} - y_i}{\Delta x}$ OR (3) $3\dfrac{y_{i+1} - y_{i-1}}{2\Delta x}$

(choose one of the options)

(1) $D = D + \dfrac{3}{\Delta x} \quad L = L - \dfrac{3}{\Delta x}$

(2) $D = D - \dfrac{3}{\Delta x} \quad R = R + \dfrac{3}{\Delta x}$

(3) $R = R + \dfrac{3}{2\Delta x} \quad L = L - \dfrac{3}{2\Delta x}$ $\rightarrow$ does not change D